Transition to phase synchronization in coupled periodically driven chaotic pendulums

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We have studied the transition to phase synchronization in the system of two coupled periodically driven pendulums. For the case of identical units, the coupled system has an infinite number of invariant subspaces. The synchronization-desynchronization transition is at the blowout bifurcation which coincides with the hyperchaos-chaos transition. On-off intermittency and intermingled basins of attraction can be observed close to this transition. For the case of nonidentical pendulums, the synchronization-desynchronization transition occurs far beyond the hyperchaos-chaos transition. The basin structure and the statistics of the accompanying intermittency are different from those for identical units.

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I. INTRODUCTION

Chaos means that two trajectories starting from slightly different initial conditions will separate exponentially with increasing time [1]. Recently, it has been shown that trajectories of such chaotic systems can be synchronized if they are properly coupled together [2]. Initially, the main interest of almost all researchers was focused on the case of coupled identical chaotic units [2–7]. Interesting phenomena such as on-off intermittency [3], riddled basins [4–7], and unstable dimension variability [8] are found near the transition to the synchronized state. Recently, the case of coupled nonidentical units [9–11] has attracted the attention of researchers due to the fact that parameter mismatch and stochastic perturbations are inevitable in real physical experiments and technical applications. Among such work, Rosenblum, Pikovsky, and Kurths showed the effect of phase synchronization of weakly coupled self-sustained chaotic oscillators [10]. Unlike other types of synchronization, it is already achieved when the coupling is extremely weak, and in some cases has no threshold.

In this paper, we study a system of two coupled chaotic pendulums. The behavior of a simple pendulum with velocity-dependent damping and periodic driving force has merged as one of the prototype model systems commonly employed in the investigation of chaotic dynamics [13]. The investigation derives additional motivation from the well known isomorphism of the driven pendulum to current biased Josephson junctions [14]. In the current study, two such periodically driven pendulums on the chaotic state are coupled together. The aim of this study is twofold. First, we want to study phase synchronization in coupled *nonautonomous* systems; this will serve as a complement to the study of phase synchronization of autonomous chaotic oscillators. Second, we want to address the issue of the synchronization-desynchronization transition of coupled *nonidentical* systems, in particular the statistics of the characteristic time for the intermittency accompanying this transition. It is found that, for a strong enough coupling, for both identical and nonidentical units, two pendulums can achieve a synchronized state satisfying the condition $|\theta_1-\theta_2|<\text{const}$ [10]. The transition to synchronization is accompanied by an intermittency where long periods of synchronized segments are occasionally interrupted by short epochs of phase slipping events. The statistics of the characteristic time of this intermittency for coupled nonidentical units is different from that of the on-off intermittency for coupled identical units, while similar to that of the eyelet intermittency accompanying the phase synchronization of autonomous systems [10]. The paper is organized as follows. In Sec. II, the coupled pendulum model studied is presented. Characteristics of the synchronization-desynchronization transition for the case of identical units are presented in Sec. III. In Sec. IV, the transition for the case of nonidentical units is studied. Finally, we outline the main results of the current study and discuss the relation between this transition and phase synchronization for autonomous systems.

II. THE MODEL OF COUPLED PENDULUMS

We use a model of two coupled chaotic pendulums with velocity-dependent damping, harmonic forcing, and an additional external constant torque. The equations describing the motion of the two pendulums are

$$\ddot{\theta}_1+\frac{1}{b_1}\dot{\theta}_1+\sin\theta_1=a_1^0+a_1\sin\omega_1 t+c\sin(\theta_2-\theta_1), \quad (1)$$

$$\ddot{\theta}_2+\frac{1}{b_2}\dot{\theta}_2+\sin\theta_2=a_2^0+a_2\sin\omega_2 t+c\sin(\theta_1-\theta_2), \quad (2)$$

where θ_i are the angular deviations of point masses from the vertical line, a_i^0 represent the strengths of the external torques, a_i and ω_i are the strengths and frequencies of the external periodic forces, c is the coupling strength, and all variables are in dimensionless form.

III. SYNCHRONIZATION TRANSITION OF TWO IDENTICAL CHAOTIC PENDULUMS

Here we set $a_1=a_2=0.78$, $\omega_1=\omega_2=0.62$, and $b_1=b_2=4.14$. In this case of two identical pendulums coupled together, the system has a number of infinite invariant subspaces $|\theta_1-\theta_2|=2n\pi$ where n is an integral. With variation of the coupling strength, the transverse stabilities of attractors on these invariant subspaces change correspondingly

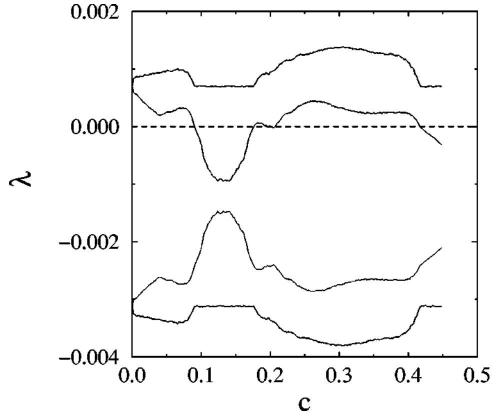

FIG. 1. Variation of LEs of coupled identical pendulums with respect to the coupling strength *c*. Zero points of the second largest LE are at $c=0.09$, 0.176, and 0.418.

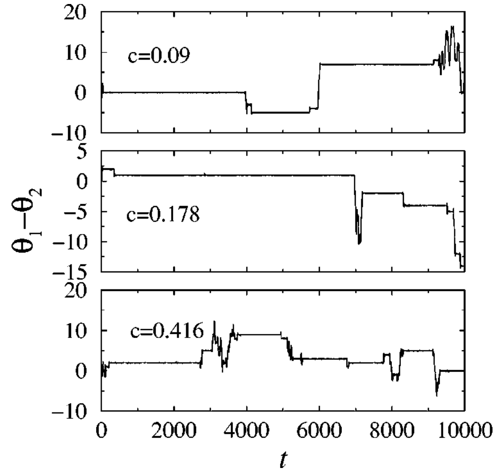

FIG. 3. The temporal evolution of the phase difference between two pendulums.

[4]. Near the point of the bifurcation from the transversely stable state to the transversely unstable one, called the blowout bifurcation [6], interesting phenomena such as on-off intermittency [3], (globally or locally) riddled basins [4–7], and unstable dimension variability [8] can be observed.

In Fig. 1, we plot the Lyapunov exponents (LEs) of the coupled system with respect to the coupling strength *c*. For $0.09<c<0.176$ and $c>4.18$, there is only one positive Lyapunov exponent in the coupled system. This means that the motions of the two pendulums are totally synchronized in these regimes.

In Fig. 2, the basin of attraction for one of the invariant states $\theta_1=\theta_2$ is shown. Here the coupling strength is $c=0.175$ in the synchronous regime. An ensemble of 300×300 points in the region $0<\theta_1<2\pi, 0<\theta_2<2\pi$ with $\dot{\theta}_1=0$ and $\dot{\theta}_2=0$ is used to construct the plot. If a point leads to the state $\theta_1=\theta_2$ after a transient of 10 000 time steps, it is left blank. Otherwise, it is denoted by a black dot. It can be seen that black dots and the blank set are interwoven together in a very complex way, and black dots penetrate the invariant subspace $\theta_1=\theta_2$ in a dense set of points. We say the basin of attraction for the attractor on the invariant subspace $\theta_1=\theta_2$ is riddled by those of other attractors. Actually, the basin of any one of the infinite number of synchronous states is riddled by those of others. This extreme kind of riddled basin is called an intermingled one in the original paper of Alexander *et al.* [4]. Here we have intermingled basins among an infinite number of attractors.

Outside the synchronous regime, the invariant subspace is no longer transversely stable. A typical trajectory stays for a long period of time near one of the invariant subspaces and occasionally escapes from it and switches to the vicinities of other invariant subspaces. The wandering of the trajectory among the infinite number of invariant subspaces gives the intermittent evolution of the phase difference between two pendulums. The phase difference $\theta_1-\theta_2$ for some cases of coupling is shown in Fig. 3. Here the on-off intermittency is of the form of switching among an infinite number of invariant states. In other words, it has an infinite number of ''off'' state in contrast to the conventional case where there is only one ''off'' state [3]. The distribution of the duration of the laminar phase, which is defined as the time duration between two phase slips, is of the universal form for on-off intermittency [3] $p(\tau)\sim l^{-1.5}\exp(-\tau/\tau_*)$. Variation of the mean length of the laminar phase with the coupling strength *c* is of the form $\langle\tau\rangle\sim(c-c_{cr})^{-1}$.

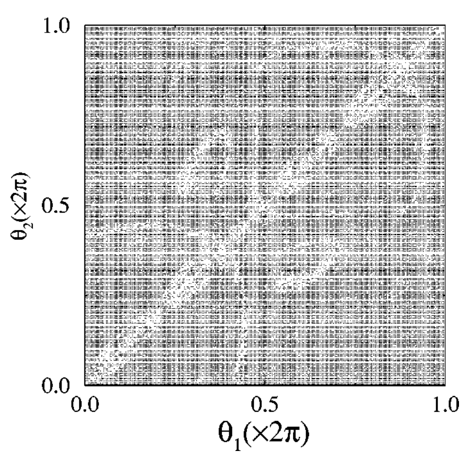

FIG. 2. The intermingled basin of the attractor $\theta_1=\theta_2$ with $c=0.175$.

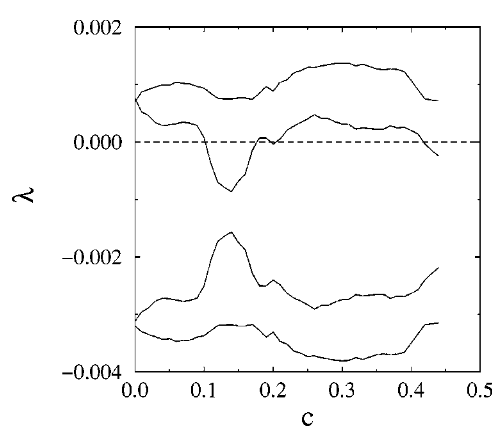

FIG. 4. Variation of LEs of two coupled nonidentical pendulums with respect to the coupling strength *c*. Zero points of the second LE are at $c=0.102$, 0.177, and 0.418.

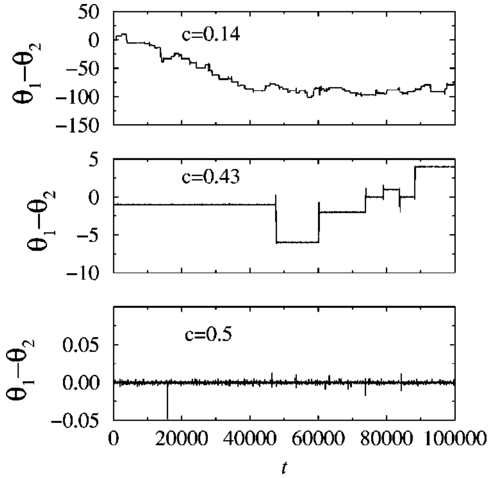

FIG. 5. The temporal evolution of the phase difference $\theta_1-\theta_2$.

IV. SYNCHRONIZATION TRANSITION OF TWO NONIDENTICAL CHAOTIC PENDULUMS

Here we use the parameter setting $a_1=a_2=0.78$, $\omega_1=\omega_2=0.62$, $b_1=4.14$, and $b_2=4.10$, i.e., with a difference in damping coefficients of the two pendulums. Without coupling, both pendulums are in chaotic states. With increasing coupling strength c, one of the two positive Lyapunov exponents becomes negative (see Fig. 4). For the case of two coupled identical pendulums, this hyperchaos-chaos transition coincides with the synchronization-desynchronization transition. About the transition, as was shown above, on one side, on-off intermittency can be observed; on the other side, a riddled basin appears. What will happen in the system of coupled nonidentical pendulums?

From numerical simulations, we found that, even beyond the hyperchaos-chaos transition, one can still observe phase slipping in the temporal evolution of the phase difference $\theta_1-\theta_2$ (see Fig. 5). Long periods of nearly phase locked states are occasionally interrupted by short epochs of quick phase slipping. Unlike the on-off intermittency in the hyperchaotic state with two positive LEs, here the intermittent state has only one positive LE. With further increase in the coupling strength, phase slipping events become rarer and rarer, and as the coupling strength is increased beyond a certain threshold the two pendulums reach the phase locked state satisfying the condition $|\theta_1-\theta_2-2n\pi|<$const.

Attractors and their basins of attraction are shown in Fig. 6. As the system is in the phase synchronized regime, it has an infinite number of attractors satisfying $|\theta_1-\theta_2-2n\pi|<$const. Unlike the case of identical pendulums coupled together, where the basins of attraction of these attractors are intermingled, here the basins of attraction are not riddled. These attractors and their basins are well separated. As can be seen in Fig. 6(a), one can find an open set to contain the attractor. With decreasing coupling strength, the basin boundary comes closer to the attractor at some points. When they coalesce, some ''channels'' appear at these points. This provides the possibility of phase slips during which the phase changes by $\pm 2n\pi$. Now the trajectory wanders among these former attractors and all of these attractors merge into a large one. It has been shown [7] that, in systems possessing an invariant subspace, the basin of attraction for a chaotic attractor in this subspace may be riddled if certain periodic orbits embedded in this chaotic attractor are transversely unstable while the whole chaotic attractor is transversely stable on average. In general, periodic orbits lose their transverse stability through the pitchfork bifurcation [4,7]. In the case of two nonidentical pendulums coupled together, there is no invariant subspace. The pitchfork bifurcation is rendered imperfect (a saddle-node bifurcation) [11]. It is conjectured that it is just this saddle-node bifurcation that opens the channel among basins of equivalent attractors. Due to the ergodicity of the chaotic state, once a channel is opened, a trajectory starting anywhere in the phase plane can go through this channel to other ''attractors.'' This leads to phase slip events. So the phase synchronization-desynchronization transition of two coupled nonidentical systems should correspond to the riddling bifurcation in the case of coupled identical systems. The bubbling bifurcation is far away from the hyperchaos-chaos transition. Thus phase slips can be observed in the deep region of the chaotic state in contrast to the fact that on-off intermittency is only in the hyperchaos regime. This difference also has a dramatic influence on the scaling of the characteristic time of the intermittency near the transition.

In Fig. 7, the distribution of the duration of phase locked segments is presented both in semilogarithmic and log-log plots. It has a power law distribution with an exponential decay on the long duration side. Numerical fitting of data gives the exponent of the power law as -1.55. This is similar to that for the on-off intermittency of coupled identical pendulums. It is expected that, as the coupling is tuned away

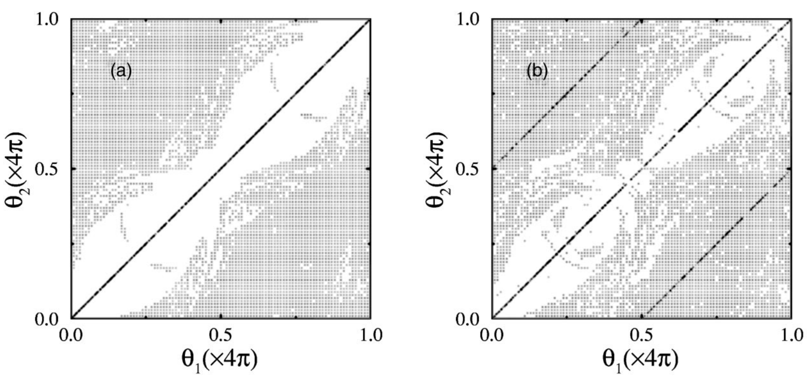

FIG. 6. Basin of attraction for coupled nonidentical pendulums with (a) $c=0.55$ in the phase synchronization regime and (b) $c=0.47$ in the phase desynchronization regime. In (a), only the attractor at $\theta_1\approx\theta_2$ is plotted. In (b), points that go first to the vicinity of $\theta_1=\theta_2$ are left blank.

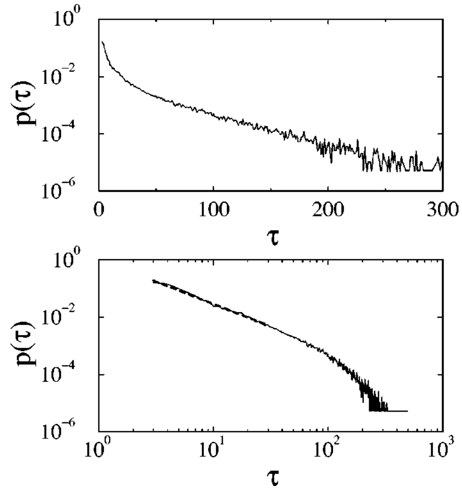

FIG. 7. Distribution of the duration of phase locked segments for the intermittency with $c=0.4$. 2×10^5 such segments are used to construct this statistic.

from the hyperchaos-chaos transition point, the region of the power law distribution will become smaller and finally disappear.

The variation of the mean length of phase locked segments with respect to the coupling strength c is plotted in Fig. 8(a). Numerical fitting shows that it is of the form $\ln\langle\tau\rangle\sim(c-c_{cr})^{-0.5}$. The average number of phase slip events during a period of 10^6 time-steps is shown in Fig. 8(b). It behaves like $\ln\langle N\rangle\sim(c-c_{cr})^{-0.5}$. This very long mean characteristic time, or, in other words, the very rare appearance of phase slips, is typical for intermittency near the attractor-repeller collision transition [12]. It makes the transition to synchronization for coupled nonidentical systems different from the blowout type of bifurcation for two coupled identical systems.

In the case studied above, the parameter mismatch between two pendulums has significant influence on the properties of the transition to synchronization. However, from the point of view of the phase dynamics, the mean frequencies (or mean angular velocities) for the two pendulums used are

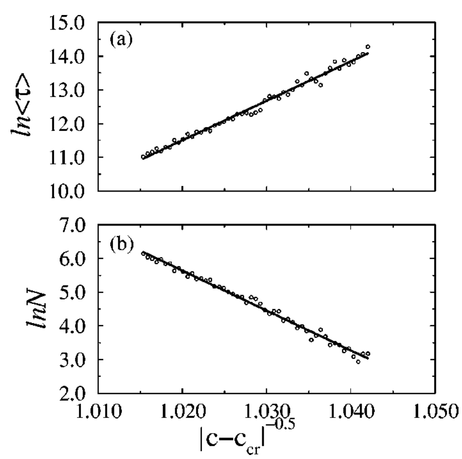

FIG. 8. (a) The mean characteristic time $\langle\tau\rangle$ of the intermittency and (b) the average number of phase slips vs the deviation of coupling strength from its critical value.

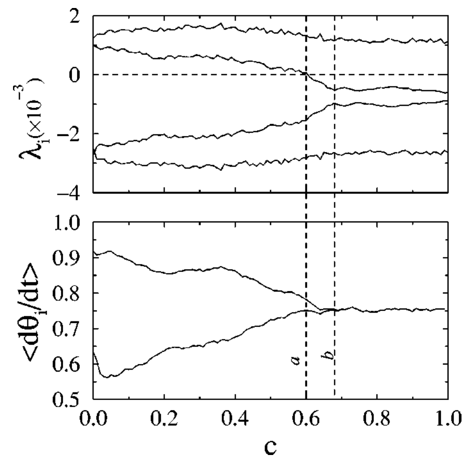

FIG. 9. Lyapunov exponents and average angular velocities for coupled pendulums with different constant external torques.

both zero, i.e., the same, irrespective of the parameter mismatch. So the synchronization of phase variables θ_i shown above is only in the sense of *phase locking* but not *frequency entrainment* [10]. Below, we will study the frequency entrainment of two chaotic pendulums with different ''frequencies,'' or different mean angular velocities $\langle\dot{\theta}\rangle$. In the language of Josephson junctions, the synchronization of two such pendulums means that, with certain suitable coupling, two different Josephson junctions with different inputs can have the same dc output. To achieve the goal of this study, we use two pendulums with different external constant torques. The parameter setting used in numerical calculations is $a_1^0=0.3$, $a_2^0=0.1$, $a_1=a_2=0.78$, $b_1=b_2=6.7$, and $\omega_1=\omega_2=0.62$.

Lyapunov exponents for such a coupled oscillator system are shown in Fig. 9. With increasing coupling strength, the second largest Lyapunov exponent becomes negative at about $c=0.6$. This is the hyperchaos-chaos transition. The variation of the average angular velocity $\langle\dot{\theta}_i\rangle$ with respect to the coupling strength c is shown in the lower frame of the same figure. From the plot, the difference between the angular velocities of the two oscillators can be seen until the coupling is about $c=0.68$, which is far beyond the hyperchaos-chaos transition. There is also a change in the LEs at this point: Below this point, the second largest LE decreases almost monotonically; beyond this point, it oscillates about a certain constant value and no fast decrease or increase happens. In our numerical calculations, it is found that frequency synchronization ($\langle\dot{\theta}_1\rangle=\langle\dot{\theta}_2\rangle$) and phase synchronization ($|\theta_1-\theta_2|<$const) are both achieved beyond the hyperchaos-chaos transition. It seems also that the two synchronizations are achieved at different coupling strengths, i.e., the frequency synchronization (at about $c=0.68$) is achieved prior to the phase synchronization. The temporal evolution of the phase difference with $c=0.7$ is shown in Fig. 10 where phase slipping events can still be seen. To determine whether the two synchronizations are really achieved at different times or simultaneously as in the case of coupled autonomous chaotic oscillators [10] still needs further numerical calculations.

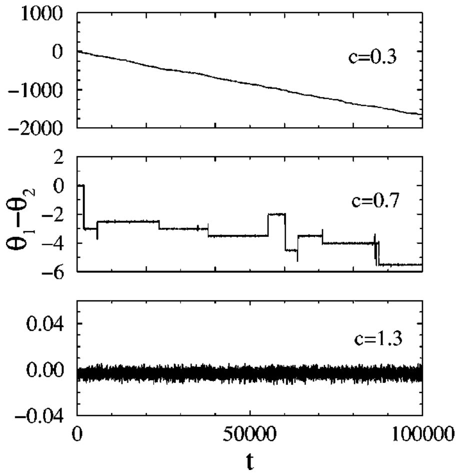

FIG. 10. The temporal evolution of the phase difference $\theta_1 - \theta_2$ with $c=0.7$.

We also studied the basin structure of the synchronized state and the statistic of the intermittency in the desynchronization regime. We obtained similar results to those for the case with different damping coefficients.

V. DISCUSSION

In this paper, we studied the transition to phase synchronization for systems of coupled identical and nonidentical chaotic oscillators with external driving.

For two coupled identical pendulums, the system has an *infinite* number of invariant subspaces. The phase synchronization-desynchronization transition is at the blowout bifurcation. In the synchronization regime, basins of attraction of the infinite number of attractors are intermingled with each other. Slightly beyond the transition, in the desynchronization regime, on-off intermittency in the phase difference between two oscillators can be observed. For two nonidentical pendulums, the transition to synchronization is far beyond the hyperchaos-chaos transition. Below the transition, in the synchronization regime, basin boundaries for the infinite number of attractors are far away from attractors. The intermittency in the phase difference for the slightly desynchronized state has a very long characteristic time.

From the point of view of bifurcation of strange attractors, the transition to synchronization for coupled identical units is a pitchfork'' bifurcation of the strange chaotic attractor on the synchronous subspace. For the case of coupled nonidentical units, the pitchfork bifurcation is smeared into a saddle-node bifurcation. Due to the ergodicity, after the saddle-node bifurcation (actually a saddle-repeller bifurcation [12]) of the least stable periodic orbit, the strange chaotic attractor becomes unstable and expands its size to include the repeller. This is different from the case for coupled identical units, where, due to the singularity of the invariant subspace, after the loss of transverse stability of the least stable periodic orbit, there is only a change in the basin of attraction (from normal to riddled) while the attractor itself is untouched until the blowout bifurcation. It is just this difference that makes the statistics for intermittencies accompanying the two transitions different.

Compared with the phase synchronization for coupled autonomous systems, the synchronized state here is reached at a much larger coupling strength. For the case of the Rössler system, the phase synchronized system is in a hyperchaotic state with two positive Lyapunov exponents. For the case studied here, the synchronized state is always in the chaotic regime with only one positive Lyapunov exponent.

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